Risk Aversion and Uncertainty in European Sovereign Bond markets

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Abstract: Risk aversion and uncertainty are often both at play in market price determination, but it is empirically challenging to disentangle one from the other. In this paper we set up a theoretical model particularly suited for opaque over-the-counter markets that is shown to be empirically tractable. Based on high frequency data, we thus propose an evaluation of risk aversion and uncertainty inherent to the government bond markets in the euro area between 2007 and 2011. We also examine the impact of the European Central Bank Security Market Program [SMP] implemented in May 2010 to ease the pressure on the European sovereign bond markets. We show in particular how this program has killed market uncertainty but raised risk aversion for all countries except Greece in a risk-pooling mechanism.

Key Words: Risk Aversion, Uncertainty, government bond market, Euro area.

Subject Classification: D40, D81, E58

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1. INTRODUCTION

European government bond markets in 2010 triggered interest of investors, and public authorities following the budget difficulties encountered by Greece and the threat to see this crisis spread all over Europe. The budgetary problems revealed by the end of 2009 by the Greek government launched a wave of hostility and mistrust of market participants. Unprecedented in the Euro area, the difficulties faced by Greece forced the European authorities to step in the market to restore market efficiency, liquidity and decrease upward tensions over bond rates. Indeed, the solvency of a state is based on the perception by market players of the quality of the traded bonds. In this sense, the most fragile countries are exposed to an increase in their probability of default given the increasingly degraded market conditions. Thus, understanding at the micro-level the determinants of bond rates and their liquidity is of first importance.

The Euro area countries collectively announced an intervention program on Sunday the 9th of May 2010 which consisted in two main measures. First, Euro area countries implemented the European Financial Stability Facility [EFSF]. This institution is composed of every Euro area countries and aims at providing financial support to countries in weak economic situations. The EFSF can raise funds by issuing bonds in the name of the Euro area. Second, the European Central Bank announced at the same time the beginning of market interventions on government bond segments to restore liquidity and to keep under control the efficient functioning of the market. This coupled announcements in a context of high tensions, appeared as a structural break in the markets, and impacted market participants behaviors. However, its specific impact on market efficiency, uncertainty, or risk aversion has not been explicitly analyzed in the literature.

To this prospect, we focus on the two key concepts that are risk aversion and uncertainty in the European government bond market. To adopt Knight (1921) view on risk and uncertainty, we define risk aversion as the fact that agents are reluctant to face choices with several probable outcomes. On the other hand, uncertainty (or ambiguity) would precise how numerous and probable are these different outcomes. In other words, the level of uncertainty is the degree of measurability of the possible outcomes when agents face risk. This explicit distinction between risk and uncertainty appears, in particular, in Gilboa and Schmeidler (1989) showing how decision makers may be influenced by some specific outcomes in a given set (e.g. the outcome with the lowest expected utility even though its probability is low).

In the microstructure analysis, Illeditsch (2009) or Easley and O’Hara (2010) provide key models
concerning uncertainty and how this dimension may influence market dynamics. The fact that investors perceive uncertainty is that they consider not one but several possible distributions for the value of the traded asset. In this paper, we consider that this phenomenon may be especially exacerbated during crises when market makers face a lot of contradictory information, and may become reluctant to propose narrow bid-ask spreads in order to protect themselves from efficiently informed traders. In this strand, we consider Biais (1993) model particularly suited for opaque and over the counter markets. This model framework particularly fits with the Euro area government bond secondary markets, for which scarce information on the best available prices is only displayed by screen process. We extend the model by introducing uncertainty on the shocks affecting the value of the asset, and show how this may directly influence the size of the bid-ask spreads given the risk aversion of the marginal market maker. We show the existence, under weak conditions, of a unique equilibrium for uncertainty and risk aversion, that is time-varying by exploiting volatilities of the bid and ask prices coupled with their durations. In particular, in the spirit of Domowitz and Wang (1994), we determine the bid-ask spread distribution in the context of our model and demonstrate that it can be well modeled by a non centered chi-square distribution whose moments depend on risk aversion, uncertainty and durations between quote revisions. This complements some other papers interested in bond market liquidity as Fleming (2003), Krishnamurty (2002) or Goldreich et al. (2005) for the US or Dunne et al. (2007) for European countries.

This model extension is done for several objectives: (i) to propose a theoretical model that allows for empirical tractability of market makers’ risk aversion and uncertainty on the government bond markets; (ii) to analyze the time series of the risk aversion and uncertainty measures provided by our model; (iii) to propose a reasonable family of distributions for the instantaneous quoted bid-ask spread as a function of risk aversion and uncertainty; (iv) and finally analyze the effect of the 9th of May 2010 announcement in the Euro Area government bond markets.

The paper is organized as follows. We first present our theoretical model with risk averse market makers facing uncertainty concerning the efficient value of the traded asset. In a second section we derive the properties of our model and we show, under weak conditions, how this model is empirically tractable. In particular, we derive the distribution properties of the instantaneous bid-ask spread. In section 3, we apply the model to France, Germany, Spain, Greece, Italy and Portugal government bond markets for three maturities (2,5 and 10 years) between 2007-2011 using high frequency data. We particularly focus on the effects of the SMP program. Section 4 concludes.
2. A MODEL WITH RISK AVERSION AND UNCERTAINTY

2.1. Model structure

2.1.1. Market maker population

Let consider a population of $N$ market makers perfectly competing for quotes in a single-asset market. This market is an over-the-counter [OTC] market characterized by its opacity and the fact that transaction details are not reported. However, there is a screening process of quotes to the extent that agents may observe the historical data for the best bid ($b^*_t$) and ask ($a^*_t$) quotes, in real time. This situation prevails in the European government bond markets for example. Each market maker is indexed by $i = 1, \ldots, N$, and each new quote arrival on the public screen is indexed by $t = 1, \ldots, T$. Note that we develop the model on an irregularly spaced timeline since $t$ is not the calendar time but the quotes arrivals. $N$ is assumed to be reasonably large and determined exogenously, especially when one considers a market such as the sovereign bond market. Liquidity is provided by dealers who declare themselves ready to trade at their bid and ask prices when they appear on the screen.

Each market maker, indexed by $i$, has a utility function with constant absolute risk aversion $A_i$ as

$$U_i(x_t) = -e^{-A_i x_{i,t}}, \quad \forall x_{i,t}$$ (1)

Note that for each agent, $A_i$ is constant over time, and $x_{i,t}$ denotes the time-varying wealth of agent $i$. Each market maker is endowed with cash $C_{i,t}$ that is net off a fixed cost related to market participation. Each agent has a current position in the traded asset $I_{i,t}$ such that the agent is long if $I_{i,t} > 0$ and short if $I_{i,t} < 0$. The final wealth of agent $i$ at date $t$ is

$$W_{i,t} = C_{i,t} + v_t I_{i,t}$$ (2)

with $v_t$ being the value of the asset revealed by the market.

2.1.2. The asset

In this model, we consider a single asset market. This asset has a value $v_t$ revealed by the market but is imperfectly known by agents. In fact, this value must not be taken as the true fundamental value of the asset in question. In addition, at each point in time, there is not necessarily a new
quote arrival so that information blurs over time. In other words, the larger the time between two quote arrivals on the screen, the more uncertain each agent is about $v_t$, and higher potential for private information to exist.

Let consider $z_t$ as the shock between the $t$ and $t+1$ quote arrivals for the value $v_{t+1}$ revealed by the market as

$$z_t = v_t - v_{t-1}$$

(3)

with $z_t$ following a Gaussian distribution with zero mean and variance $\sigma^2_{z_t}$. We assume $\sigma^2_{z_t}$ to be directly proportional to the observed durations between two quote issuances similar in the empirical literature with an Autoregressive conditional duration model (ACD) as in Engle and Russell (1998). Even if $z_t$ is never observed, agents build some expectations for $\sigma^2_{z_t}$ based on the expectations of the variances ($\tilde{\sigma}^2_{a_t}$ and $\tilde{\sigma}^2_{b_t}$) and covariance ($\tilde{\sigma}_{a_t,b_t}$) for $a^*_t$ and $b^*_t$ the ask and bid prices that appear on screens:

$$\sigma^2_{z_t} = \pi_t \tilde{\sigma}^2_{a^*_t} + (1 - \pi_t)^2 \tilde{\sigma}^2_{b^*_t} + \pi_t (1 - \pi_t) \tilde{\sigma}_{a^*_t,b^*_t}$$

The parameter $\pi_t$ is assumed to be the weight attributed to each price variation (at the bid and at the ask prices) and is defined as

$$\pi_t \sim N \left( \frac{1}{2}, \sigma^2_{\pi_t} \right)$$

(4)

In this setup, $\sigma^2_{\pi_t}$ reflects the uncertainty concerning the value of the asset $v_t$ revealed by the market at transaction $t$. $\sigma^2_{\pi_t}$ gives, in a sense, the extent of the set of probabilities that $\pi_t$ can reach: in other words the set of distributions that agents may expect for the shock on the value of the asset $v_t$. For example, if we assume the absence of uncertainty so that $\sigma^2_{\pi_t} = 0$, finally $\pi_t = \frac{1}{2}$ and we obtain that $\sigma^2_{z_t} = var(\frac{\Delta a^*_t + \Delta b^*_t}{2})$ the volatility of the midquote variations: in this case the variations of the midquote reveals the variations of the asset value. However, the larger $\sigma^2_{\pi_t}$, the more uncertainty we have in the market.

2.1.3. Information set and quote sequences

As already mentioned, the only piece of information agents have about the traded asset is the one given by a centralized screen. We denote $\Omega_t$ the information set a time $t$ that is common knowledge.
to any market maker as

$$\Omega_t = \{(a_u^t, b_u^t, \tau_u^t)\}_{u=1...t}$$ (5)

which comprises the past observations for declared best ask and bid prices, and the past delays in time unit between two quote arrivals $\tau^u$. In this setup, the time between two quote revisions is crucial. For example, consider an agent at time $t - 1$ who determines her optimal levels of bid and ask prices for transaction $t$, she knows that the longer she waits, the higher the uncertainty about $v_t$ is. As a consequence to be the one appearing on the screen at $t$, she needs to have the best expected bid-ask spread and needs to be the fastest. This appears in the next section when, given this setup, we derive the optimal level for the bid and ask prices.

2.2. Reservation quotes and quote revisions

2.2.1. Market expectations

Before deriving their optimal quotes, all agents build some market expectations given the available information at date $t - 1$, $\Omega_{t-1}$. The first quantity of interest is the instantaneous volatility of the ask price

$$E(\sigma_{a_t^2}^2 | \Omega_{t-1}) = \sigma_{a_{t-1}^2}$$ (6)

where both expectation for the future instantaneous volatility and the past observed instantaneous volatility under the hypothesis of rational expectations. In the empirical section, we explain in more details the estimation process of the instantaneous variances and covariance between the ask and the bid prices. Similarly, the expected instantaneous variance of the bid price and the expected instantaneous covariance of the bid and ask prices are given by

$$E(\sigma_{b_t^2}^2 | \Omega_{t-1}) = \sigma_{b_{t-1}^2}$$ (7)

and

$$E(\sigma_{a_t^2 b_t^2} | \Omega_{t-1}) = \sigma_{a_{t-1}^2 b_{t-1}^2}.$$ (8)

All these quantities are elements of the instantaneous variance covariance matrix for the ask and bid prices as
Given this evaluation of market instantaneous variances and covariances, agents infer an implicit measure of the variations of $v_t$ as a function of $\pi_t$ such that

$$E(\Sigma_t | \Omega_{t-1}) = \Sigma_{t-1} = \begin{bmatrix} \sigma_{a_t}^2 & \sigma_{a_t, b_t} \sigma_{b_t}^2 \\ \sigma_{a_t, b_t} & \sigma_{b_t}^2 \end{bmatrix}$$ (9)

Variances are increasing in durations as it is usually considered in the empirical literature of ACD models. In our theoretical framework, this is related to the price for immediacy as in Chacko et al. (2008) such that impatient market makers have to propose faster and narrower spreads than competitors to appear on the screen and this is a cost related to their impatience.

2.2.2. Reservation quotes

Reservation quotes are the bid and ask prices that make the market maker indifferent. Given the market expectations previously derived, each agent computes her reservation quotes such that she is indifferent between potentially buying the underlying asset at the bid reservation quote, selling at the ask reservation quote and doing nothing. These reservation quotes are not the optimal ones in a sense that they do not maximize the surplus in wealth that they could expect from trading with the public. However, in the case of perfect competition as it is in our case, the expected surplus is null. In other words, the aim of posting some quotes on the market is not perceived as an effective search for surplus, but more for reputation purposes. Indeed, appearing on the screen for market makers is important to signal to other market participants their presence on the trading of the asset even if this signal should not expose the market maker to excessive risk (inventory risk in our case) once her quotes are displayed.

At time $t-1$ each dealer is endowed with cash $C_i, t-1$ and has a current position in the traded asset $I_i, t-1$. In the event that she does not quote prices, her final wealth when a new quote is posted in $t$ by a competitor is supposed to be $W_{i, t}(0)$ as:

$$W_{i, t}(0) = C_i, t-1 + I_{i, t-1} v_t$$ (11)

Alternatively, facing a market buy order for a quantity $Q_t$ at the ask prices gives a final wealth
of $W_{i,t}(a_{i,t})$ as:

$$ W_{i,t}(a_{i,t}) = C_{i,t-1} + I_{i,t-1}v_t + (a_{i,t} - v_t)Q_t $$

(12)

On the contrary, if the dealer $i$ buys a quantity $Q_t$ at price $b_{i,t}$, her final wealth is $W_{i,t}(b_{i,t})$:

$$ W_{i,t}(b_{i,t}) = C_{i,t-1} + I_{i,t-1}v_t + (v_t - b_{i,t})Q_t $$

(13)

The ask reservation quote and the bid reservation quote for each agent $i$ at date $t+1$ are denoted $\bar{a}_{i,t+1}$ and $\bar{b}_{i,t+1}$ respectively such that

$$ E[U(W_{i,t}(0)) | \Omega_{t-1}, \pi_t] = E[U(W_{i,t}(\bar{a}_{i,t})) | \Omega_{t-1}, \pi_t] = E[U(W_{i,t}(\bar{b}_{i,t})) | \Omega_{t-1}, \pi_t]. $$

(14)

Given our setup, and close to the Biais (1993) model, we obtain that

$$ \bar{a}_{i,t} = \frac{A_i}{2}(Q_t - 2I_{i,t-1})E(\sigma^2_{z_t} | \Omega_{t-1}, \pi_t) + v_t $$

(15)

$$ \bar{b}_{i,t} = -\frac{A_i}{2}(Q_t + 2I_{i,t-1})E(\sigma^2_{z_t} | \Omega_{t-1}, \pi_t) + v_t $$

(16)

And the reservation bid-ask spread is equal to:

$$ S_{i,t} = \bar{a}_{i,t} - \bar{b}_{i,t} = A_i Q_t E(\sigma^2_{z_t} | \Omega_{t-1}, \pi_t) $$

(17)

Even if we do not have any information concerning the level of $v_t$, the fundamental value of the asset, the spread reflects both the risk aversion coefficient and the expected instability of the asset value. An increase in the risk aversion component $A_i$ tends to shift up the reservation ask price and exerts downward pressure on the bid price, which widens the reservation spread. Moreover, the higher $E(\sigma^2_{z_t} | \Omega_{t-1}, \pi_t)$, the larger the reservation spread. Such a phenomenon is quite intuitive since the dealer is willing to protect herself from significant variations of the asset’s value by quoting a large spread.

Assuming perfect competition in the market for posting quotes, the smallest and fastest spread appears on the screen so that

$$ S^*_t | \Omega_{t-1}, \pi_t = \min(S_{i,t}, i = 1...N) $$

(18)
such that

\[ S_t^* \mid \Omega_{t-1}, \pi_t = A^* Q_t^* \tilde{\tau}_t \left[ \pi_t^2 \sigma_{a_{t-1}}^2 + (1 - \pi_t)^2 \sigma_{b_{t-1}}^2 + 2\pi_t (1 - \pi_t) \sigma_{a_{t-1}} \sigma_{b_{t-1}} \right] \mid \Omega_{t-1}, \pi_t . \]  

(19)

Assuming that \( \pi_t \sim N(\frac{1}{2}, \sigma_{\pi_t}^2) \) we can rewrite the expected spread only conditional on \( \Omega_t \) as

\[ E(S_t^* \mid \Omega_{t-1}) = A^* Q_t^* \tilde{\tau}_t \left[ \sigma_{\pi_t}^2 \left( \sigma_{a_{t-1}}^2 + \sigma_{b_{t-1}}^2 - 2\sigma_{a_{t-1}} \sigma_{b_{t-1}} \right) + Var(\Delta M_{t-1}^*) \right] \]  

(20)

where \( Var(\Delta M_{t-1}^*) \) is the variance of the previous midquote. Equation (20) directly illustrates the impact of risk aversion and uncertainty on the expectation of the optimal spread for next quote update and thus on market liquidity. If higher risk aversion leads to a larger bid-ask spread, stronger uncertainty (large \( \sigma_{\pi_t}^2 \)) also widens the expected best spread appearing on the screen at quotation \( t \). The linear impact of \( \tilde{\tau}_t \), the duration, on expected quoted spreads is related, as we said before, to some price for immediacy as in Chacko et al. (2008) but also on the fact that the spread is increasing given that there is no quote revision: as long as agents do not have enough valuable information to revise their quotes, they stay out of the market, bid-ask spreads increase and liquidity becomes scarce.

Note that in our model, agents are not explicitly averse to uncertainty, as it is in Easley and O’Hara (2010) for example with the maximization of the minimum of the market maker utility. In our setup, agents face uncertainty as given, and this appears in the spread because they are risk averse. As a consequence, if \( A^* \), the risk aversion of the marginal market maker is nought the spread cancels even if there is uncertainty in the market which is quite a reasonable result. However, we can have some situations of zero uncertainty, which does not mean that the spread is zero but just that the midquote of the spread is truly revealing the variations of the asset value.

At first glance, this spread equation shows that several combinations of risk aversion and uncertainty may lead to the same level of spread even if the market scenarios can be quite different. For example, considering a low level of risk aversion, with high uncertainty may lead to the same spread as high risk aversion with low uncertainty. However, these two situations are not comparable at all.

However, in the next section, we show how this model is empirically tractable and discuss the weak conditions for a unique equilibrium \( (A^*, \sigma_{\pi_t}^2)_t \) characterizing both risk aversion and uncertainty, but disentangling the two different concepts for the best spreads appearing, in real-time, on the
3. EMPIRICAL TRACTABILITY OF THE THEORETICAL MODEL

There are several key quantities which need an empirical counterpart in our model. In this section, we first provide some empirical specification for these quantities, and then we show that we can obtain under weak conditions and empirical estimation of risk aversion and uncertainty at any point in time.

3.1. Variance, covariances and durations

We first consider a set of model specifications to extract market instantaneous volatilities $\sigma_{a_t^2}$, $\sigma_{b_t^2}$, and $\sigma_{a_t b_t^*}$. Our model presents two variables of interest, the ask and the bid prices observed at each quote issuance. In the present paper, a non parametric approach is adopted to facilitate the empirical tractability of the model over time.

We fairly assume that market makers anticipate what future variances will be, based on the observations of the previous price variations. The non parametric specification retained is the one developed by Barndorff-Nielsen and Shephard (2003), the so-called first order bipower variations. Therefore, the bid and ask prices vector, $(a_t, b_t)$, has a variance-covariance matrix according to the following formula:

$$
\Sigma_t = \begin{pmatrix}
\Sigma_{aa}^t & \Sigma_{ab}^t \\
\Sigma_{ba}^t & \Sigma_{bb}^t
\end{pmatrix}
$$

(21)

with

$$
\Sigma_{aa}^t = \frac{\delta}{4} \sum_{j=0}^{J} \frac{\Delta a_{t-j-1}}{\sqrt{T_{t-j-1}}} \frac{\Delta a_{t-j}}{\sqrt{T_{t-j}}} 
$$

(22)

$$
\Sigma_{bb}^t = \frac{\delta}{4} \sum_{j=0}^{J} \frac{\Delta b_{t-j-1}}{\sqrt{T_{t-j-1}}} \frac{\Delta b_{t-j}}{\sqrt{T_{t-j}}} 
$$

(23)

$$
\Sigma_{ab}^t = \frac{\delta}{4} \sum_{j=0}^{J} \frac{\Delta a_{t-j-1}}{\sqrt{T_{t-j-1}}} \frac{\Delta b_{t-j-1}}{\sqrt{T_{t-j-1}}} \frac{\Delta a_{t-j}}{\sqrt{T_{t-j}}} + \frac{\Delta b_{t-j}}{\sqrt{T_{t-j}}} \\
- \frac{\Delta a_{t-j-1}}{\sqrt{T_{t-j-1}}} - \frac{\Delta b_{t-j-1}}{\sqrt{T_{t-j-1}}} \frac{\Delta a_{t-j}}{\sqrt{T_{t-j}}} - \frac{\Delta b_{t-j}}{\sqrt{T_{t-j}}}
$$

(24)

where $J$ is the number of lag price variations considered by the market maker to infer volatility, and
\[ \delta = \frac{J}{\tau} \] is a scaling parameter. This gives our instantaneous variance covariance matrix. We also need an expected duration \( \tilde{\tau}_t \) to extract \((A^*, \sigma_z^2)\) at each transaction. Durations are assumed to follow an AR\(p\) process such that we have some clusters in durations, with the alternation of periods of high frequency revisions with low frequency revisions.

### 3.2. Uncertainty and risk aversion at equilibrium

In this section, we derive, under some weak assumptions, the unique solution for \((A^*, \sigma_z^2)\) for each new quotation. As seen before in equation (20), the expected spread conditional on \( \Omega_{t-1} \) can be written

\[
E(S_t^* | \Omega_{t-1}) = A^* Q_t^* \tilde{\tau}_t \left[ \sigma_{\pi_t}^2 (\sigma_{a_t-1}^2 + \sigma_{b_t-1}^2 - 2\sigma_{a_t-1} \sigma_{b_t-1}) + \frac{1}{4} (\sigma_{a_t-1}^2 + \sigma_{b_t-1}^2 + 2\sigma_{a_t-1} \sigma_{b_t-1}) \right].
\] (25)

Or

\[
E(S_t^* | \Omega_{t-1}) = A^* Q_t^* \tilde{\tau}_t \left[ \sigma_{\pi_t}^2 \text{Var}(a_{t-1}^* - b_{t-1}^*) + \text{Var}(\frac{a_{t-1}^* + b_{t-1}^*}{2}) \right].
\] (26)

For the moment we exclude the quantity effect. By this assumption, we assume that for a given bond the implicit quantity for which bid and ask prices are quoted by market makers is constant over the whole sample. The volatility of the expected optimal spread is

\[
\text{Var}(S_t^* | \Omega_{t-1}) = A^{*2} \tilde{\tau}_t^2 \left[ 2 \left[ \text{Var}(a_{t-1}^* - b_{t-1}^*) \right]^2 \sigma_{\pi_t}^4 + (\sigma_{a_t-1}^2 - \sigma_{b_t-1}^2)^2 \sigma_{\pi_t}^2 \right].
\] (27)

**Proposition 1.** Given equations (25) and (27), there exists a unique positive equilibrium \((A^*, \sigma_z^2)\) at any quote update \(t\) if

\[
\text{Var}(a_{t-1}^* - b_{t-1}^*) = 0
\] (28)

or if

\[
\text{Var}(a_{t-1}^* - b_{t-1}^*) \neq 0 \quad \text{and} \quad \frac{\text{Var}(S_t^* | \Omega_{t-1})}{[E(S_t^* | \Omega_{t-1})]^2} < 2
\]

so that the model is empirically tractable for any spread distribution whose coefficient of variation is lower than \(\sqrt{2}\). Proof of proposition 1 is reported in appendix A.

From the proposition stated above, we can derive another proposition.
Proposition 2. Assuming that $\pi_t \sim N\left(\frac{1}{2}, \sigma_{\pi_t}^2\right)$ and $\text{Var}(a_{t-1}^* - b_{t-1}^*) \neq 0$, then the distribution of the spread $S_t$ conditional on $\Omega_{t-1}$ is a Gaussian distribution with mean $\mu_{S_t}$ and variance $\sigma_{S_t}^2$ equal to

$$\mu_{S_t} = A^* \tilde{\tau}_t \sigma_{a_{t-1}^* - b_{t-1}^*}^2$$

$$\sigma_{S_t}^2 = A^* \tilde{\tau}_t^2 (\sigma_{a_{t-1}^*}^2 - \sigma_{b_{t-1}^*}^2)^2 \sigma_{\pi_t}^2$$

(29) (30)

If $\text{Var}(a_{t-1}^* - b_{t-1}^*) \neq 0$, its distribution is a non centered and non reduced $\chi^2$ distribution with one degree of freedom and with a parameter related to the mean of the random variable $\pi_t$, $\lambda_t = \frac{1}{4} \left( \frac{\sigma_{a_{t-1}^*}^2 - \sigma_{b_{t-1}^*}^2}{\text{Var}(a_{t-1}^* - b_{t-1}^*)} \right)^2$ and therefore with mean $\mu_{S_t}$ and variance $\sigma_{S_t}^2$ equal to

$$\mu_{S_t} = A^* \tilde{\tau}_t \left[ \text{Var}(a_{t-1}^* - b_{t-1}^*)(\sigma_{\pi_t}^2 + \lambda_t) + \sigma_{b_{t-1}^*}^2 - (\sigma_{a_{t-1}^*}^2 - \sigma_{b_{t-1}^*}^2)^2 \right]$$

(31)

and

$$\sigma_{S_t}^2 = 2A^* \tilde{\tau}_t^2 \text{Var}(a_{t-1}^* - b_{t-1}^*) \left[ \sigma_{\pi_t}^4 + 2\lambda_t \sigma_{\pi_t}^2 \right].$$

(32)

Its probability density function is, $\forall s \in \mathbb{R}_+$,

$$f_{S_t}(s) = \frac{1}{A^* \tilde{\tau}_t \text{Var}(a_{t-1}^* - b_{t-1}^*)\sigma_{\pi_t}^2} \frac{h^{1}_{\lambda_t} \left( \frac{\text{Var}(a_{t-1}^* - b_{t-1}^*)s + A^* \tilde{\tau}_t (\sigma_{a_{t-1}^*}^2 - \sigma_{b_{t-1}^*}^2)}{A^* \tilde{\tau}_t \text{Var}(a_{t-1}^* - b_{t-1}^*)^2 \sigma_{\pi_t}^2} \right)}{h^{1}_{\lambda_t} \left( \frac{\text{Var}(a_{t-1}^* - b_{t-1}^*)}{\sigma_{\pi_t}^2} \right)}$$

(33)

where $h^{1}_{\lambda_t}$ is the probability density function of a non centered chi-square distribution with one degree of freedom. Proof of proposition 2 is reported in appendix B.

In our theoretical framework, we model the distribution of the instantaneous bid-ask spread at each new quote arrival. However, we should keep in mind that our model takes into account both bid-ask spread in itself and duration. These two liquidity components are intrinsically linked and both have an impact on the price discovery process: as explained before, a larger expected duration will increase market makers' expectations in terms of the volatility of the asset value revealed by the market, leading dealers to widen their spreads. Therefore, this time dimension has a sizeable effect on the expected optimal quoted bid-ask spread.

From the previous results, we can notice that both the expectation and the variance of the optimal bid-ask spread increase with risk aversion and uncertainty. They are also convex functions in the uncertainty. All these theoretical findings show us as expected that these two components have a tendency to widen the expected bid-ask spread but also the spectrum of possible outcomes...
To determine the risk aversion, the uncertainty and then the distribution of the expected optimal spreads at any point in time we apply a two-step procedure. First we estimate $\Sigma_t$ given our specification of equation (21). Then by solving the quadratic expression derived in Appendix A at each point in time, we obtain $(A^*, \sigma^2_\pi)_t$.

4. EMPIRICAL APPLICATION

4.1. Dataset and Stylized facts on the euro government bond market

To derive the historical paths of our measures of risk aversion and uncertainty, we use high frequency data provided by Thomson Reuters Tick History. We mainly focus our analysis on 6 European government bonds (France, Germany, Greece, Italy, Portugal, Spain) for three maturities (2, 5 and 10 years). The data cover the period from January 4th, 2007 to February 1st, 2011. The Thomson Reuters Tick History database contains date-and-time stamped bid-ask quotes. At each new quote issuance, the best ask (bid) price, which appears on the screen corresponds to the lowest (highest) price offered by market-makers. Therefore, having only access to best prices leads us to call upon the previous model based on a competitive market.

Before trying to extract relevant information from this dataset, we conduct some prior data processing in order to remove non-valid quotations. Observations whose either ask or bid price is equal to zero, bid-ask spread is negative or explosive due to reporting errors are deleted. Moreover, we reduce our sample by only taking quotes which are issued between 8:00 and 18:00 GMT. Indeed, even if government bonds are traded on over-the-counter markets and therefore transactions of such assets may take place all day long, numbers of observed quote issuances are only significant when European markets are open. We can reasonably assume that quotes issued for European bonds on the American or Asian markets, which are relatively scarce, would add no relevant value and could only disturb the running of our analysis. Finally, we remove days which count less than two hundred observations as they could lead to misleading findings.

Table 1 reports some descriptive statistics from the raw data of the different government bonds studied. Major differences appear between mean and median statistics, which indicates the presence of extreme values (confirmed by large standard deviations) for both bid-ask spreads and durations. This confirms that sovereign bond markets are marked by episodes of high volatilities that can result from uncertainty and/or risk aversion. We can notice some heterogeneity across countries
Table 1: Descriptive statistics of observed bid-ask spreads and durations (time between two consecutive quote issuances)

Notes: The table reports summary statistics for the different European government bonds studied. From high frequency data, mean, median and standard deviation are computed for the whole period considered (from the start of the sample to August, 8th 2010).

<table>
<thead>
<tr>
<th></th>
<th>Spread</th>
<th>Duration (in seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Median</td>
</tr>
<tr>
<td>DE2YTRR</td>
<td>0.042 0.04</td>
<td>0.084</td>
</tr>
<tr>
<td>DE5YTRR</td>
<td>0.052 0.04</td>
<td>0.069</td>
</tr>
<tr>
<td>DE10YTRR</td>
<td>0.051 0.04</td>
<td>0.096</td>
</tr>
<tr>
<td>FR2YTRR</td>
<td>0.05 0.03</td>
<td>0.06</td>
</tr>
<tr>
<td>FR5YTRR</td>
<td>0.059 0.03</td>
<td>0.077</td>
</tr>
<tr>
<td>FR10YTRR</td>
<td>0.094 0.05</td>
<td>0.148</td>
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<td>GR10YTRR</td>
<td>0.499 0.06</td>
<td>0.906</td>
</tr>
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</table>
and maturities for the quoted bid-ask spreads. In particular, countries recently under stress in the
context of the sovereign bond crisis have unsurprisingly recorded larger than usual bid-ask spreads.
This feature is clear looking at mean bid-ask spreads but also in the distributions presented in the
appendix C. The different humps observed may indicate the presence of time-varying distributions.
Durations also show significant differences. Quotes for the French and German bonds are globally
revised less frequently than the ones of other countries (see Figures 1 and 2 in the appendix C).

One first striking feature is that we detect an impact of maturities on spreads and durations
as in Elton and Green (1998). We particularly find that spreads increase with maturities whereas
durations decrease with maturities. Such a phenomenon seems to be rather counterintuitive. Indeed,
both bid-ask spreads and durations between two quote issuances can be seen as liquidity indicators.
They should both rise when market conditions deteriorate: when an asset is assumed less liquid, risk
averse and uncertain market makers widen the spread offered by way of hedging and transactions
(and therefore quote revisions) occur less often. Nevertheless, an other explanation can be brought
to the fore: presence of smaller durations does not mean that the asset in question is traded more
frequently. This can reflect an high uncertainty and/or risk aversion environment in which market
makers revise their quotes more often in order to protect themselves from potential large price
variations or informed traders.

We display in appendix C the historical evolutions of the daily median bid-ask spreads (Figure 3).
Even if we observe some heterogeneity across countries and maturities regarding quoted spreads,
the end of the period has been globally marked by a surge in this liquidity indicator. We must
however point out that spreads of France and Germany have broadened at the outbreak of the
crisis and then shrunk maybe due to some flight-to-quality effects. The historical paths of the daily
median durations between two consecutive quote issuances are also provided in appendix C (Figure
4). Similar comments that the ones mentioned about the quoted bid-ask spreads can be made. In
addition, for the very recent period and for all maturities, durations reached unprecedented low levels
especially for Spain, Italy, Portugal, Greece and to a lesser extent for Germany. We may suppose
that in a period of crisis the high level of uncertainty leads market maker to more frequently revise
their quotes.

Looking at intraday data, hourly patterns for spreads exhibit two groups of countries (Figures 5
and 6). For France and Germany, spreads slightly decrease during the day and then finally rise by
the end of the day. For the second group of countries, the spreads have a tendency to narrow and
this feature fastens from 4 p.m. Market makers for this bunch of countries are more likely to take some risks at the end of the day. On the other hand, duration patterns are slightly homogeneous across countries with a significant surge starting around 3 p.m.

4.2. The European government bond crisis

The model previously presented and its empirical counterpart are particularly suited for the analysis of the SMP program of the ECB. To our knowledge, this is the first paper looking at the impact of the SMP program in such a way. The graphical results are therefore displayed into two parts, before and after April 2010, i.e. one month before the collective announcement of May the 9th, 2010.

From equation 25, we directly see that risk aversion is key to the definition of the spread. Indeed, if the $A$ coefficient is nought, then the spread is zero, whatever is the level of uncertainty in the market. On the other side, even if uncertainty is nought, it does not mean that the spread is zero but the market efficiently reveals the asset value at the midquote.

Figure 7 in the appendix D exhibits the evolutions of the daily median of risk aversion and uncertainty for the 2-year maturity. Firstly, we can notice that the variability of these measures are quite high but contained. Market makers’ risk aversion towards Germany is the lowest one. For Italy, Greece and Portugal, some spikes in this measure appear during the summer-autumn 2009 until the beginning of 2010. On the other hand, for France, we observe high risk aversion only during summer-autumn 2009. In terms of uncertainty, France and Germany present the lowest levels, which means that the price discovery process is the most efficient for them. Italy, Spain and Portugal are intermediate cases while Greece appears to be efficient to a lesser extent. Uncertainty regarding Spain, Italy and Portugal reaches high levels during the summer 2007 whereas the one towards France and Italy only spikes at the beginning of 2008.

We can notice that for Spain, Greece, Italy and Portugal, bid-ask spreads soared during October 2008 and the beginning of 2010. However, the increase during 2008 was smaller than the one observed during the sovereign bond crisis. In general, for the 5-year and 10-year maturities, in Figures 8 and 9, we observe a significant decrease in risk aversion during October 2008 with the surge of the Lehman Brothers crisis. This remains true for the most of the countries until spring 2009. Most of the countries have taken advantage of some flight-to-quality and liquidity effects. From April 2009, we notice that in Portugal, Greece, Italy and Spain, risk aversion progressively
increases without reaching the pre-crisis levels and to finally surge at the beginning of 2010 for Portugal and Greece.

Regarding uncertainty, we globally observe that the 5-year maturity bonds are the ones with the highest coefficients compared to the other maturities. We observe high levels of uncertainty during the 2007 crisis for the 2-year maturity whereas the same phenomenon is noticed for the 5 and 10-year maturities during the 2008 crisis. This indicates that agents have expected some long-lasting effects of the 2008 crisis compared to the 2007 turmoil.

Given the deteriorating situation on the Greek bond market and to annihilate any contagion phenomena to other countries, the Euro area Governing Council opted to implement the Security Market Program [SMP]. This decision was suddenly taken in the weekend of May 9th 2010 to be effective on Monday 10th. This implied some striking changes in our indicators, reflecting a strong reversal in market makers’ sentiment on every bond segment.

Following May 10th, the communication at the Euro area level has killed global uncertainty for Greece, Portugal, Italy and Spain (Figures 10,11 and 12). On the contrary, for France and Germany, uncertainty remains at some similar levels or even rise at some point in time after the implementation of the program. This is in line with the risk-pooling mechanism at the Euro area level, that is also revealed by the dynamics of the risk aversion measure. Indeed, risk aversion for Portugal, Italy, Spain and France (only for the 2 and 5-year maturities) has increased after the announcement of the SMP whereas it has decreased for Greece. Consequently, the collective support beyond this mechanism has lowered the threat of dramatic isolated situations in the Euro area.

5. CONCLUSION

Market participants during crises usually face high risk and a lot of information, potentially contradictory, that may complicate the pricing of some assets. In this paper, we introduce, in a model for opaque OTC markets, the uncertainty dimension, that may make market makers reluctant to participate in the price discovery process. Even if participants are not explicitly ambiguity averse we show that the level of uncertainty, coupled with risk aversion, has a direct impact on the bid-ask spreads and thus can deteriorate market liquidity. In particular, the high levels of uncertainty and risk aversion during crises tend to give higher probability to very large bid-ask spreads. Thus, thanks to our model, we are able to extract at each quote arrival instantaneous bid-ask spread whose shape directly depends on the degrees of uncertainty and risk aversion that markets encounter. Beyond
the theoretical analysis of uncertainty, the proposed model presents an empirical tractability that allows for the analysis of the euro area bond market crisis. In particular, we empirically analyzed risk aversion and uncertainty on a pool of six countries (France, Germany, Greece, Spain, Italy and Portugal) for three maturities (2, 5 and 10 years) and had a closer look on the impact of announcement of the SMP program in May 2010.

Our main conclusions are as follows. First, the historical evolutions of our risk aversion and uncertainty measures reveal periods of severe tensions that occurred in the sovereign bond market between 2007 and the beginning of 2010. As expected, they mainly coincide with the subprime crisis and the Lehman Brother’s bankruptcy episode. Our model allows us to have a better understanding of what happened in these markets in terms of liquidity. For instance, during the summer 2007, market makers quoting short-term maturity European government bonds were both risk averse and uncertain regarding the outcome of the financial turmoil. On the other hand, medium and long-term maturities seem to have not been affected by the subprime crisis, which may indicate that markets did not expect the crisis would last. This noticeable market agents’ behavior disappeared when Lehman Brother defaulted, with a surge in both risk aversion and uncertainty towards peripheral countries for every maturity.

In terms of monetary policy implications, this paper provides some insights to better understand the impact of the several measures that the ECB has adopted so far to ease off on the pressure in these markets. The SMP program has succeeded in restoring market efficiency by killing market uncertainty. The resolution to not allow the default of any country in the Euro area and the ability to directly intervene to provide funding appeared as a really strong and collective commitment. However, the announcement has the adverse consequences to rise risk aversion over all Euro area members due to the risk-pooling mechanism beyond this commitment, except for Greece which has taken advantage of this decision.
References:


Knight F., 1921, Risk, Ambiguity, and Profit, *Houghton, Mifflin, Boston, MA*.

6. APPENDIX

Appendix A.

Proof of Proposition 1. From the assumption that \( \pi_t \sim N(\frac{1}{2}, \sigma^2_{\pi_t}) \) and the formula (17) and (18), we get:

\[
E(S^*_t | \Omega_{t-1}) = E(A^*\tilde{\pi}_t \left[ \pi_t^2 \sigma^2_{\pi_{t-1}} + (1 - \pi_t)^2 \sigma^2_{\gamma_{t-1}} + 2\pi_t (1 - \pi_t) \sigma_{\pi_{t-1}} \nu_{t-1} \right] | \Omega_{t-1}).
\]

(34)

\[
= A^*\tilde{\pi}_t [E[\pi^2_t Var(a^*_{t-1} - b^*_{t-1}) | \Omega_{t-1}] + 2E[\pi_t (\sigma^2_{\pi_{t-1}} \nu_{t-1} - \sigma^2_{\gamma_{t-1}}) | \Omega_{t-1}] + \sigma^2_{\gamma_{t-1}}].
\]

As the kurtosis coefficient of a Gaussian distribution equals 3, \( Var(a^*_{t-1} - b^*_{t-1}) | \Omega_{t-1} \) and \( \sigma^2_{\gamma_{t-1}} \) are negative and with the assumption \( \mu_{\pi_t} = \frac{1}{2} \) then \( \pi_t \sim N(\mu, \sigma^2_{\pi_t}) \), we have:

\[
E(S^*_t | \Omega_{t-1}) = A^*\tilde{\pi}_t [Var(a^*_{t-1} - b^*_{t-1})^2 Var(\pi^2_t | \Omega_{t-1}) + 4(\sigma^2_{\gamma_{t-1}} - \sigma^2_{\pi_{t-1}})^2 Var(\pi_t | \Omega_{t-1}) + 4(\sigma^2_{\gamma_{t-1}} - \sigma^2_{\pi_{t-1}}) Cov(\pi_t, \pi^2_t | \Omega_{t-1}) + (\sigma^2_{\pi_{t-1}} - \sigma^2_{\gamma_{t-1}})^2 \sigma^2_{\pi_t}].
\]

(35)

As the kurtosis coefficient of a Gaussian distribution equals 3, \( E((\pi_t - E(\pi_t))^4) = 3\sigma^4_{\pi_t} \).

Moreover, the skewness coefficient is equal to 0, \( E((\pi_t - E(\pi_t))^3) = 0 \).

Then, after some mathematical computation, as \( E(\pi^4_t) = 3\sigma^4_{\pi_t} - 6\sigma^2_{\pi_t} \mu^2 - 3\mu^4 + 4\mu E(\pi^2_t) \) and \( E(\pi^3_t) = \mu^3 + 3\mu \sigma^2_{\pi_t} \), we obtain:

\[
Var(\pi^2_t | \Omega_{t-1}) = 2\sigma^4_{\pi_t} + 4\sigma^2_{\pi_t} \mu^2 \text{ and } Cov(\pi_t, \pi^2_t | \Omega_{t-1}) = \sigma^2_{\pi_t}.
\]

If we have \( Var(a^*_{t-1} - b^*_{t-1}) = 0 \) (then \( Var(a^*_{t-1} - b^*_{t-1}) \neq 0 \) and \( \sigma^2_{\pi_{t-1}} - \sigma^2_{\gamma_{t-1}} \neq 0 \) otherwise neither the ask nor the bid price move between \( t-2 \) and \( t-1 \), which is impossible) then from 34 and 35, \( \Lambda^* = \frac{4E(S^*_t | \Omega_{t-1})}{\pi_t Var(a^*_{t-1} - b^*_{t-1})} \) and \( \sigma^2_{\pi_t} = \frac{Var(S^*_t | \Omega_{t-1})}{A^*2\pi_t^2(\sigma^2_{\pi_{t-1}} - \sigma^2_{\gamma_{t-1}})^2} \).

From the system of two equations 34 and 35 of the first and second moment of the expected quoted spread at \( t \) conditional on the information set \( \Omega_{t-1} \), we can derive the value of the risk.
aversion coefficient $A^*$ and the variance of the coefficient $\pi_t$ (measure of uncertainty). By extracting
an expression of $A^*$ from Equation 34 as follows

$$A^* = \frac{E(S_t^* | \Omega_{t-1})}{\hat{\pi}_t [\sigma_{\pi_t}^2 Var(a_t^* - b_t^*) + \frac{1}{2} Var(a_t^* + b_t^*)]}.$$ 

we can obtain a quadratic equation for $\sigma_{\pi_t}^2$. Indeed from Equation 35 we have

$$Var(S_t^* | \Omega_{t-1}) = \left(\frac{E(S_t^* | \Omega_{t-1})}{\hat{\pi}_t [\sigma_{\pi_t}^2 Var(a_t^* - b_t^*) + \frac{1}{2} Var(a_t^* + b_t^*)]}\right)^2 \times (36)$$

so that the quadratic equation for $\sigma_{\pi_t}^2$ is the following:

$$(Var(a_t^* - b_t^*)^2 Var(S_t^* | \Omega_{t-1}) - 2Var(a_t^* - b_t^*)^2 E(S_t^* | \Omega_{t-1})^2) \sigma_{\pi_t}^4 + (37)$$

$$= 0.$$ 

We denote

$$a = Var(a_t^* - b_t^*)^2 Var(S_t^* | \Omega_{t-1}) - 2Var(a_t^* - b_t^*)^2 E(S_t^* | \Omega_{t-1})^2$$

$$b = \frac{1}{2}Var(a_t^* - b_t^*)Var(a_t^* + b_t^*)Var(S_t^* | \Omega_{t-1}) - (\sigma_{\pi_t}^2)^2 E(S_t^* | \Omega_{t-1})^2$$

$$c = \frac{1}{16}Var(a_t^* + b_t^*)^2 Var(S_t^* | \Omega_{t-1})^2$$

The discriminant $\Delta$ of this quadratic equation is equal to

$$\Delta = b^2 - 4ac$$

$$= \left[\frac{1}{2}Var(a_t^* - b_t^*)Var(a_t^* + b_t^*)Var(S_t^* | \Omega_{t-1}) - (\sigma_{\pi_t}^2)^2 E(S_t^* | \Omega_{t-1})^2\right]^2$$

$$-\frac{1}{4}Var(a_t^* + b_t^*)^2 Var(S_t^* | \Omega_{t-1}) \times$$

$$(Var(a_t^* - b_t^*)^2 Var(S_t^* | \Omega_{t-1}) - 2Var(a_t^* - b_t^*)^2 E(S_t^* | \Omega_{t-1})^2)$$
Assume that the condition \( \frac{\text{var}(S_{t}^{*}|\Omega_{t-1})}{\text{Var}(S_{t}^{*}|\Omega_{t-1})} < 2 \) is verified, then the second term of the discriminant is strictly positive and there exist two real roots of the quadratic equation. Besides, as the two roots are respectively equal to \( x_{1} = -\frac{b_{t}}{2a} \) and \( x_{2} = -\frac{b_{t}+\sqrt{\Delta}}{2a} \), we can compute

\[
x_{1}x_{2} = \frac{b_{t}^{2} - \Delta}{4a^{2}} = \frac{c}{a} = \frac{1}{16} \text{Var}(a_{t-1}^{*} + b_{t-1}^{*})^{2} \text{Var}(S_{t}^{*}|\Omega_{t-1})
\]

\[
= \frac{1}{16} \text{Var}(a_{t-1}^{*} - b_{t-1}^{*})^{2} \text{Var}(S_{t}^{*}|\Omega_{t-1}) - 2 \text{Var}(a_{t-1}^{*} - b_{t-1}^{*})^{2} \text{Var}(S_{t}^{*}|\Omega_{t-1})^{2}
\]

The product of the two roots is negative if \( \frac{\text{var}(S_{t}^{*}|\Omega_{t-1})}{\text{Var}(S_{t}^{*}|\Omega_{t-1})} < 2 \) as assumed. Under such assumption, there exist one unique positive real root and one unique negative real root. It follows that we have a unique positive solution for the uncertainty \( \sigma_{t_{1}}^{2} \), and necessarily a unique solution for \( A^{*} \) the risk aversion coefficient. Q.E.D.

**Appendix B.**

*Proof of Proposition 2.* From equation 19 and assuming \( \text{Var}(a_{t-1}^{*} - b_{t-1}^{*}) \neq 0 \) (the other case is straightforward and quite simple), we can rewrite the optimal bid-ask spread \( S_{t}^{*} \) (whose the random part comes from \( \pi_{t} \) and which is conditional on \( \Omega_{t-1} \): we do not precise it at each step of the proof):

\[
S_{t}^{*} = A^{*} \hat{\pi}_{t} \left[ \frac{\pi_{t}^{2} \sigma_{a_{t-1}^{*}}^{2} + (1 - \pi_{t})^{2} \sigma_{b_{t-1}^{*}}^{2} + 2 \pi_{t}(1 - \pi_{t}) \sigma_{a_{t-1}^{*}b_{t-1}^{*}}}{\text{Var}(a_{t-1}^{*} - b_{t-1}^{*}) \pi_{t}^{2} + 2 \pi_{t}(\sigma_{a_{t-1}^{*}}^{2} \pi_{t}^{2} \sigma_{a_{t-1}^{*}b_{t-1}^{*}} + \sigma_{b_{t-1}^{*}}^{2})} \right].
\]

As \( Y_{t} = \pi_{t} + \frac{\sigma_{a_{t-1}^{*}b_{t-1}^{*}} - \frac{\sigma_{b_{t-1}^{*}}^{2}}{\text{Var}(a_{t-1}^{*} - b_{t-1}^{*})}}{\text{Var}(a_{t-1}^{*} - b_{t-1}^{*})} \) follows a Gaussian distribution whose mean is \( \frac{\sigma_{a_{t-1}^{*}b_{t-1}^{*}}}{\text{Var}(a_{t-1}^{*} - b_{t-1}^{*})} \) and
variance $\sigma_{\tau t}^2$, $Z_t = \left( \pi_t + \frac{(\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^* b_{\tau t-1}^*)}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right)^2$ follows a non centered and non reduced $\chi^2$ distribution with one degree of freedom. According to the formula of a non centered $\chi^2$ distribution and after some calculations, its mean and variance are

$$\mu_{Z_t} = \sigma_{\tau t}^2 + \lambda_t$$

$$\sigma_{Z_t} = 2\sigma_{\tau t}^4 + 4\lambda_t \sigma_{\tau t}^4$$

where $\lambda_t = \frac{1}{4} \left( \frac{\sigma_{\tau t-1}^2 - \sigma_{\tau t-1}^*}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right)^2$.

Therefore, $S_t^*$ follows a non centered and non reduced $\chi^2$ distribution with one degree of freedom and with mean and variance equal to

$$\mu_{S_t} = A^* \tau_t \left[ \text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*) (\sigma_{\tau t}^2 + \lambda_t) + \sigma_{\tau t-1}^2 - (\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^2)^2 \right]$$

$$\sigma_{S_t}^2 = 2A^* \tau_t^2 \text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)^2 \left[ \sigma_{\tau t}^4 + 2\lambda_t \sigma_{\tau t}^2 \right].$$  \hspace{1cm} (38)

In addition, for $\forall s \in \mathbb{R}_+$,

$$F_{S_t}(s) = P(S_t \leq s)$$

$$= P \left( \left( \pi_t + \frac{(\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^* b_{\tau t-1}^*)}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right)^2 \leq \frac{(\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^* b_{\tau t-1}^*)^2}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)^2} + \frac{\sigma_{\tau t-1}^2}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right)$$

$$\leq \frac{A^* \tau_t \text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)}{s} \left( \pi_t + \frac{(\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^* b_{\tau t-1}^*)}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right)^2 \leq \frac{\pi_t^2 \sigma_{\tau t}^2}{A^* \tau_t \text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)^2} \left( \frac{(\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^* b_{\tau t-1}^*)^2}{\sigma_{\tau t}^4 \text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)^2} - \frac{\sigma_{\tau t-1}^2}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right)$$

$$= P \left( \left( \pi_t + \frac{(\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^* b_{\tau t-1}^*)}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right)^2 \leq \frac{(\sigma_{\tau t-1}^* b_{\tau t-1}^* - \sigma_{\tau t-1}^* b_{\tau t-1}^*)^2}{\sigma_{\tau t}^4 \text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)^2} + \frac{\sigma_{\tau t-1}^2}{\text{Var}(a_{\tau t-1}^* - b_{\tau t-1}^*)} \right).$$
As $\frac{Z_t}{\sigma^2_{\pi t}}$ follows a non centered $\chi^2_1$ distribution, $\forall s \in \mathbb{R}_+$ we have,

$$f_{S_t}(s) = \frac{dF_{S_t}(s)}{ds}$$

$$= \frac{1}{A^*\tilde{\tau}_t \text{Var}(a^*_{t-1} - b^*_{t-1}) \sigma^2_{\pi t}} \times$$

$$h^1_{\chi^2_{\pi c}} \left( \frac{\text{Var}(a^*_{t-1} - b^*_{t-1}) s + A^*\tilde{\tau}_t (\sigma^2_{a^*_{t-1} - b^*_{t-1}} - \sigma^2_{a^*_{t-1} - b^*_{t-1}})}{A^*\tilde{\tau}_t \text{Var}(a^*_{t-1} - b^*_{t-1})^2 \sigma^2_{\pi t}} \right)$$

where $h^1_{\chi^2_{\pi c}}$ is the probability density function of a non centered chi-square distribution with one degree of freedom. Q.E.D.
FIG. 1 Histograms of the daily median bid-ask spreads.
FIG. 2 Histograms of the daily median durations.
FIG. 3 Evolutions of the daily median bid-ask spreads.
FIG. 4 Evolutions of the daily median durations.
FIG. 5 Hourly patterns of the quoted bid-ask spreads.
FIG. 6 Hourly patterns of the durations.
FIG. 7 Evolutions of the daily median extracted risk aversion coefficients and uncertainty measures for the 2-year-to-maturity government bonds before April 2010.
FIG. 8 Evolutions of the daily median extracted risk aversion coefficients and uncertainty measures for the 5-year-to-maturity government bonds before April 2010.
FIG. 9 Evolutions of the daily median extracted risk aversion coefficients and uncertainty measures for the 10-year-to-maturity government bonds before April 2010.
FIG. 10 Evolutions of the daily median extracted risk aversion coefficients and uncertainty measures for the 2-year-to-maturity government bonds from April to August 2010.
FIG. 11 Evolutions of the daily median extracted risk aversion coefficients and uncertainty measures for the 5-year-to-maturity government bonds from April to August 2010.
FIG. 12 Evolutions of the daily median extracted risk aversion coefficients and uncertainty measures for the 10-year-to-maturity government bonds from April to August 2010.